Deriving the inverse square law from radiative transfer equations

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ABSTRACT

The radiative transfer equation (RTE) is the fundamental equation of the radiative transfer theory and one of more important theoretical tools in biomedical optics for describing light propagation in biological tissues. The RTE assumes that the refractive index of the medium is constant and the ray divergence is zero. These assumptions limit its range of applicability. To eliminate this drawback three new RTE have been proposed recently. Obviously, those equations must be carefully studied and compared. With that aim we solve the standard RTE and the new radiative transfer equations for the specific case of a time-independent isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index. The solution to this problem is the well-known inverse square law of geometrical optics. We show that only one of those equations gives solutions consistent with the inverse square law for the irradiance, due to its ability to model non-negligible ray divergence near a point source.

Keywords: radiative transfer equation

1. INTRODUCTION

The radiative transfer equation (RTE) is the fundamental equation of the radiative transfer theory and one of the more important theoretical tools in biomedical optics and related fields.[1-3] However, the model leading to the radiative transfer equation assumes that the refractive index of the medium is constant and that the divergence of the rays is always zero.[1] Both assumptions limit the range of physical situations for which the RTE is valid.[4] Recently more general radiative transfer equations have been derived. Ferwerda[5] and Khan and Jiang[6] derived a radiative transfer equation for media with spatially varying refractive index (Ferwerda-Khan-Jiang RTEvri) that permits spatial variations of refractive index and nonzero ray divergence. Tualle and Tinet[7] derived a radiative transfer equation for media with spatially varying refractive index (Tualle-Tinet RTEvri), which assumes that the ray divergence is always zero. Martí-López et al.[4] derived a radiative transfer equation for media with spatially varying refractive index (Martí-Bouza-Hebden-Arridge-Martínez RTEvri) which differs from the Ferwerda-Khan-Jiang RTEvri in the expression for the ray divergence. Obviously, all these new radiative transfer equations for media with spatially varying refractive index (RTEvri) must be carefully compared and tested. We have attempted to do this by applying these equations to problems of light propagation with well-known solutions and comparing the solutions obtained from each one. Such problems can be considered as benchmarks for comparing those equations. It should be pointed out that benchmarking is a widely used tool in numerical analysis, neural networks and other disciplines for comparing algorithms, methods of solution and mathematical models.

One of the basic assumptions of the radiative transfer theory is that wave properties of light and related phenomena can be ignored. As a consequence light propagation is described as a process of energy transport along ray trajectories.[1] This description is consistent with that fundamental to geometrical optics. Accordingly, the results of the radiative transfer theory for non-absorbing non-amplifying non-scattering linear media should coincide with the predictions of geometrical

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optics for some cases. This circumstance favors the creation of benchmark problems for comparing radiative transfer equations. Specifically the calculation of the irradiance using radiative transfer equations for a time-independent isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index should yield the well-known inverse square law of geometrical optics.

In this paper we solve the RTE, the Ferwerda-Khan-Jiang RTEvri, the Tualle-Tinet RTEvri and the Martí-Bouza-Hebden-Arridge-Martínez RTEvri for a time-independent isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index. We show that the RTE, the Ferwerda-Khan-Jiang RTEvri and the Tualle-Tinet RTEvri fail to yield the inverse square law of geometrical optics for irradiance. By contrast, the Martí-Bouza-Hebden-Arridge-Martínez RTEvri gives a solution consistent with the inverse square law for the irradiance. However, all these equations correctly yield the inverse square law for the modulus of the radiant current density vector. The success of the Martí-Bouza-Hebden-Arridge-Martínez RTEvri may be explained by its facility to model non-negligible ray divergence near a point source.

2. RADIATIVE TRANSFER EQUATIONS AND RAY DIVERGENCE

The standard RTE has the form:[3]

$$\frac{n(\mathbf{r})}{c}\frac{\partial L(\mathbf{r},\boldsymbol{\Omega},t)}{\partial t}+\boldsymbol{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}L(\mathbf{r},\boldsymbol{\Omega},t)+\left[\mu_a(\mathbf{r})+\mu_s(\mathbf{r})\right]L(\mathbf{r},\boldsymbol{\Omega},t)$$
$$=\mu_s(\mathbf{r})\int_{4\pi}\theta(\mathbf{r},\boldsymbol{\Omega},\boldsymbol{\Omega}')L(\mathbf{r},\boldsymbol{\Omega},t)\,\mathrm{d}\omega'+\varepsilon(\mathbf{r},\boldsymbol{\Omega},t), \tag{1}$$

where $L(\mathbf{r},\boldsymbol{\Omega},t)$ is the radiance at a point $\mathbf{r}=x_1\hat{\mathbf{x}}_1+x_2\hat{\mathbf{x}}_2+x_3\hat{\mathbf{x}}_3$ in the direction $\boldsymbol{\Omega}=\Omega_1\hat{\mathbf{x}}_1+\Omega_2\hat{\mathbf{x}}_2+\Omega_3\hat{\mathbf{x}}_3$, $\boldsymbol{\Omega}$ is the unit vector tangential to the ray, $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ are the unit vectors of the Cartesian coordinate system, $\mu_a(\mathbf{r})$ and $\mu_s(\mathbf{r})$ are the absorption and scattering coefficients, respectively, $\theta(\mathbf{r},\boldsymbol{\Omega},\boldsymbol{\Omega}')$ is the normalized scattering function, $\boldsymbol{\Omega}'$ is the direction of propagation of an incident ray, $d\omega'$ is a differential of solid angle, and $\varepsilon(\mathbf{r},\boldsymbol{\Omega},t)$ is the distribution of sources. When operator $\nabla_{\mathbf{r}}$ acts on the radiance $L(\mathbf{r},\boldsymbol{\Omega},t)$ it does not affect the ray direction $\boldsymbol{\Omega}$.

In equation (1) the refractive index n is constant and ray divergence is assumed to be zero, i.e. $\nabla_{\mathbf{r}}\cdot\boldsymbol{\Omega}=0$ [1]. The Ferwerda-Khan-Jiang RTEvri and the Martí-Bouza-Hebden-Arridge-Martínez RTEvri have the form:[4, 6]

$$\frac{n(\mathbf{r})}{c}\frac{\partial L(\mathbf{r},\boldsymbol{\Omega},t)}{\partial t}+\boldsymbol{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}L(\mathbf{r},\boldsymbol{\Omega},t)+\left[\nabla_{\mathbf{r}}\cdot\boldsymbol{\Omega}(\mathbf{r})\right]L(\mathbf{r},\boldsymbol{\Omega},t)+\nabla_{\mathbf{r}}\ln n(\mathbf{r})\cdot\nabla_{\boldsymbol{\Omega}}L(\mathbf{r},\boldsymbol{\Omega},t)$$
$$=-[\mu_a(\mathbf{r})+\mu_s(\mathbf{r})]L(\mathbf{r},\boldsymbol{\Omega},t)+\mu_s(\mathbf{r})\int_{4\pi}L(\mathbf{r},\boldsymbol{\Omega}',t)\theta(\mathbf{r},\boldsymbol{\Omega},\boldsymbol{\Omega}')\mathrm{d}\omega+\varepsilon(\mathbf{r},\boldsymbol{\Omega},t), \tag{2}$$

where operator $\nabla_{\boldsymbol{\Omega}}$ acts on the direction of the ray. Those equations contain two new terms $\left[\nabla_{\mathbf{r}}\cdot\boldsymbol{\Omega}(\mathbf{r})\right]L(\mathbf{r},\boldsymbol{\Omega},t)$ and $\nabla_{\mathbf{r}}\ln n(\mathbf{r})\cdot\nabla_{\boldsymbol{\Omega}}L(\mathbf{r},\boldsymbol{\Omega},t)$, and differ from each other in the expressions for the ray divergence. For the Ferwerda-Khan-Jiang RTEvri it has the form: [5]

$$\nabla_{\mathbf{r}}\cdot\boldsymbol{\Omega}(\mathbf{r})=\frac{1}{\Omega_x}\frac{\partial\ln n(\mathbf{r})}{\partial x}+\frac{1}{\Omega_y}\frac{\partial\ln n(\mathbf{r})}{\partial y}+\frac{1}{\Omega_z}\frac{\partial\ln n(\mathbf{r})}{\partial z}-3\nabla_{\mathbf{r}}\ln n(\mathbf{r})\cdot\boldsymbol{\Omega}(\mathbf{r}), \tag{3}$$

and for the Martí-Bouza-Hebden-Arridge-Martínez RTEvri it has the form[4]

$$\nabla_{\mathbf{r}}\cdot\boldsymbol{\Omega}(\mathbf{r})=\mu_d(\mathbf{r})-\boldsymbol{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}\ln n(\mathbf{r}), \tag{4}$$

The divergence coefficient $\mu_d(\mathbf{r})$ occurring in (4) is given by:

$$\mu_d(\mathbf{r}) = \frac{\nabla_{\mathbf{r}}^2 \Lambda(\mathbf{r})}{n(\mathbf{r})}, \tag{5}$$

where $\Lambda(\mathbf{r})$ is the eikonal of geometrical optics.[8] Meanwhile the Tualle-Tinet RTEvri assumes a zero ray divergence and, in comparison with the RTE (1), has two new terms $-2[\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}} \ln n(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)$ and $\nabla_{\mathbf{r}} \ln n(\mathbf{r})\cdot\nabla_{\mathbf{\Omega}} L(\mathbf{r},\mathbf{\Omega},t)$:[7]

$$\begin{aligned}&\frac{n(\mathbf{r})}{c}\frac{\partial L(\mathbf{r},\mathbf{\Omega},t)}{\partial t}+\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}L(\mathbf{r},\mathbf{\Omega},t)-2[\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}\ln n(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)+\nabla_{\mathbf{r}}\ln n(\mathbf{r})\cdot\nabla_{\mathbf{\Omega}}L(\mathbf{r},\mathbf{\Omega},t)\\&=-[\mu_a(\mathbf{r})+\mu_s(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)+\mu_s(\mathbf{r})\int_{4\pi}L(\mathbf{r},\mathbf{\Omega}',t)\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')\mathrm{d}\omega+\varepsilon(\mathbf{r},\mathbf{\Omega},t).\end{aligned} \tag{6}$$

From a comparison of the RTE (1), the Ferwerda-Khan-Jiang RTEvri [equations (2) and (3)], the Martí-Bouza-Hebden-Arridge-Martínez RTEvri [equations (2), (4) and (5)] the Tualle-Tinet RTEvri (6), it follows that:

1. The RTE and the Tualle-Tinet RTEvri lack a term describing the ray divergence.

2. The difference between the Ferwerda-Khan-Jiang RTEvri and the Martí-Bouza-Hebden-Arridge-Martínez RTEvri is given only by the expression for the ray divergence.

From the above points it follows that one of the basic features to be compared in those radiative transfer equations is their treatment of ray divergence. Since the terms describing ray divergence do not contain the absorption or scattering coefficients we may perform this comparison using the specific case where $\mu_a(\mathbf{r}) = \mu_s(\mathbf{r}) = 0$, i.e. using a simple problem with nonzero ray divergence where a known solution from geometrical optics is available for benchmarking. Specifically, we choose one of the simplest problems in geometrical optics, which is the radiation of light by a time-independent isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index. Its solution is given by the well-known inverse square law.[8]

3. THE INVERSE SQUARE LAW

According to the inverse square law of geometrical optics the irradiance emitted by a time-independent isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index, located at the origin of a Cartesian coordinate system has the form:[8]

$$I(r) = \frac{c_1}{r^2}, \tag{7}$$

where $r = |\mathbf{r}|$ and c_1 is a positive constant.

A similar expression for the modulus of the Poynting vector can be derived from formula (7) using the relationship between the irradiance and the Poynting vector in the incoherent case.[8] Therefore the radiant current density vector $\mathbf{J}(\mathbf{r})$ must be of the form:

$$\mathbf{J}(r) = J(r)\hat{\mathbf{r}}, \tag{8}$$

$$J(r) = \frac{c_2}{r^2}, \tag{9}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ and c_2 is a positive constant. In expressions (8) and (9) we exploit the symmetric properties of this problem. In the following sections we try to derive expressions (7) and (9) from the RTE, the Ferwerda-Khan-Jiang RTEvri, the Tualle-Tinet RTEvri and the Martí-Bouza-Hebden-Arridge-Martínez RTEvri.

4. THE INVERSE SQUARE LAW AND THE RTE

The time-independent RTE (1) for an infinite non-absorbing non-amplifying non-scattering linear medium with constant refractive index takes the form:

$$\mathbf{\Omega}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} L(\mathbf{r}, \mathbf{\Omega}) = \varepsilon(\mathbf{r}, \mathbf{\Omega}), \tag{10}$$

where $\varepsilon(\mathbf{r}, \mathbf{\Omega})$ is the source distribution. Now we assume that the radiance $L(\mathbf{r}, \mathbf{\Omega})$ and the power distribution $\varepsilon(\mathbf{r}, \mathbf{\Omega})$ can be expanded in series of spherical harmonics of the form:

$$L(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi} I(\mathbf{r}) + \frac{3}{4\pi} \mathbf{\Omega} \cdot \mathbf{J}(\mathbf{r}) + R_L(\mathbf{r}, \mathbf{\Omega}), \tag{11}$$

$$\varepsilon(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi} \varepsilon_0(\mathbf{r}) + \frac{3}{4\pi} \mathbf{\Omega} \cdot \boldsymbol{\varepsilon}_1(\mathbf{r}) + R_\varepsilon(\mathbf{r}, \mathbf{\Omega}), \tag{12}$$

where $I(\mathbf{r})$, $\mathbf{J}(\mathbf{r})$, $\varepsilon_0(\mathbf{r})$ and $\boldsymbol{\varepsilon}_1(\mathbf{r})$ are the irradiance, the radiant current density vector, the zeroth and first moments of de power distribution, respectively, given by the expressions:

$$I(\mathbf{r}) = \int_{4\pi} L(\mathbf{r}, \mathbf{\Omega}) \mathrm{d}\omega, \tag{13}$$

$$\mathbf{J}(\mathbf{r}) = \int_{4\pi} L(\mathbf{r}, \mathbf{\Omega}) \mathbf{\Omega} \mathrm{d}\omega, \tag{14}$$

$$\varepsilon_0(\mathbf{r}) = \int_{4\pi} \varepsilon(\mathbf{r}, \mathbf{\Omega}) \mathrm{d}\omega, \tag{15}$$

$$\boldsymbol{\varepsilon}_1(\mathbf{r}) = \int_{4\pi} \varepsilon(\mathbf{r}, \mathbf{\Omega}) \mathbf{\Omega} \mathrm{d}\omega, \tag{16}$$

and $R_L(\mathbf{r}, \mathbf{\Omega})$ and $R_\varepsilon(\mathbf{r}, \mathbf{\Omega})$ are remainders grouping the terms of second order or higher of the series (11) and (12), respectively. Further we assume that the P_1 approximation is valid.[3] It means that $R_L(\mathbf{r}, \mathbf{\Omega}) = 0$ and $R_\varepsilon(\mathbf{r}, \mathbf{\Omega}) = 0$.

For an isotropic point source located at the origin of coordinates we have:

$$\varepsilon_0(\mathbf{r}) = \frac{P_0}{4\pi} \delta(\mathbf{r}), \quad \boldsymbol{\varepsilon}_1(\mathbf{r}) = \mathbf{0}, \tag{17}$$

where P_0 is the radiant flux (power) and $\delta(\mathbf{r})$ is the Dirac delta distribution. Substituting expansion (11) into expression (10), multiplying the resulting expression by $\mathbf{\Omega}$, integrating the result over 4π sr and using equation (17) we obtain,

$$\nabla_{\mathbf{r}} I(r) = 0 \Rightarrow I(r) = \text{constant}. \tag{18}$$

Therefore, the expression for the irradiance (18) derived from the RTE is not consistent with the inverse square law for the irradiance (7). To derive the expression for the radiant current density is simpler. First we add the term

$\left[\nabla_{\mathbf{r}}\cdot\mathbf{\Omega}(\mathbf{r})\right]L(\mathbf{r},\mathbf{\Omega})$ to the left-hand side of equation (10). This operation does not alter equation (10) because the RTE assumes that $\nabla_{\mathbf{r}}\cdot\mathbf{\Omega}(\mathbf{r})=0$.[1] Second, we apply the vector identity

$$\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}L(\mathbf{r},\mathbf{\Omega})+\left[\nabla_{\mathbf{r}}\cdot\mathbf{\Omega}(\mathbf{r})\right]L(\mathbf{r},\mathbf{\Omega})=\nabla_{\mathbf{r}}\cdot\left[\mathbf{\Omega}(\mathbf{r})L(\mathbf{r},\mathbf{\Omega})\right], \tag{19}$$

and integrate over 4π sr . Then we obtain the equation:

$$\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r})=P_0\delta(\mathbf{r}). \tag{20}$$

Applying the Gauss theorem and the radial symmetry of the problem we obtain:

$$J(r)=\frac{P_0}{4\pi r^2}. \tag{21}$$

Therefore we find that while the RTE fails to yield the inverse square law for the irradiance (7), it does correctly yield the inverse square law for the modulus of radiant current density (9).

5. THE INVERSE SQUARE LAW, THE FERWERDA-KHAN-JIANG RTEvri AND THE TUALLE-TINET RTEvri

For a medium of constant refractive index the ray divergence (3) and the terms containing the gradient of refractive index vanish. Consequently, the Ferwerda-Khan-Jiang RTEvri and the Tualle-Tinet RTEvri reduce to equation (10) which leads to expressions (18) and (21). Therefore they also fail to yield the inverse square law for the irradiance (7), but yield the inverse square law for the modulus of the radiant current density vector (9).

6. THE INVERSE SQUARE LAW AND THE MARTÍ-BOUZA-HEBDEN-ARRIDGE-MARTÍNEZ RTEvri

For the conditions of this problem equation (2) reduces to:

$$\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}L(\mathbf{r},\mathbf{\Omega})+\left[\nabla_{\mathbf{r}}\cdot\mathbf{\Omega}(\mathbf{r})\right]L(\mathbf{r},\mathbf{\Omega})=\frac{P_0}{4\pi}\delta(\mathbf{r}). \tag{22}$$

Applying the vector identity (19) and integrating over 4π sr we obtain equation (20) and its solution (21). Therefore the Martí-Bouza-Hebden-Arridge-Martínez RTEvri correctly yields the inverse square law for the radiant current density (9). To derive the irradiance we need to calculate the ray divergence. For an isotropic point source located at the coordinate origin in an infinite non-absorbing non-amplifying non-scattering linear medium of constant refractive index, the eikonal, the ray divergence (4) and the divergence coefficient (5) are:[3]

$$\Lambda(\mathbf{r})=nr, \tag{23}$$

$$\mu_d(\mathbf{r})=\frac{2}{r}, \tag{24}$$

$$\nabla_{\mathbf{r}}\cdot\mathbf{\Omega}=\frac{2}{r}. \tag{25}$$

Note that the nonzero ray divergence (25) does not alter the symmetry of this problem. Substituting expression (25) and the spherical harmonic expansion (11) into equation (19) we obtain:

$$\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}I(\mathbf{r})+3\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathbf{r}}\left[\mathbf{\Omega}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})\right]+\frac{2}{r}I(\mathbf{r})+\frac{6}{r}\mathbf{\Omega}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})=P_0\delta(\mathbf{r}). \tag{26}$$

Multiplying equation (26) by $\mathbf{\Omega}$, integrating over 4π sr and writing the result in spherical coordinates we get

$$\frac{d}{dr}I(r) = -\frac{6}{r}J(r). \tag{27}$$

Substituting expression (21) into equation (27) and solving the resulting equation we obtain

$$I(r) = \frac{3P_0}{4\pi r^2}, \tag{28}$$

which is consistent with the inverse square law of geometrical optics of the irradiance (7). Note that the expression for the ray divergence (25) played a key role in deriving result (28).

CONCLUSIONS

The well-known problem of optical radiation diverging from an isotropic point source in an infinite non-absorbing non-amplifying non-scattering linear medium was used as a benchmark for comparing the RTE, the Ferwerda-Khan-Jiang RTEvri, the Tualle-Tinet RTEvri and the Martí-Bouza-Hebden-Arridge-Martínez RTEvri. The solution to that problem is the inverse square law of geometrical optics.[8] Our results can be summarized as follows:

1. The RTE, the Ferwerda-Khan-Jiang RTEvri and the Tualle-Tinet RTEvri yield a constant irradiance, and consequently fail to recover the inverse square law (7).

2. The compared radiative transfer equations give the same solution for the radiant current density vector. That solution is consistent with the inverse square law for the Poynting vector as derived from geometrical optics.

3. Only the Martí-Bouza-Hebden-Arridge-Martínez RTEvri yields a solution consistent with the inverse square law of the irradiance (7). This is a consequence of a divergence term included in the equation.

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