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An investigation of the ability of modified radiative transfer equations to accommodate laws of geometrical optics

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Abstract

The radiative transfer equation (RTE) and its approximations are widely used for describing light propagation in biological tissues. However, the RTE is valid for media with constant refractive indices, an assumption that does not hold in many practical situations. Recently three RTEs for media with spatially varying refractive index (RTEvri) have been proposed to eliminate that limitation. In this paper we test the RTE and the new RTEvris, applying them to solve two problems of geometrical optics with well-known solutions. We show that only one of those equations gives solutions consistent with the laws of geometrical optics due to its ability to model the effect of spatially varying refractive index and non-negligible ray divergence. This process allows us to determine which RTEvri provides the best description of light propagation in turbid media with spatially varying refractive index and a link between the radiative transfer theory and geometrical optics.

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Keywords: Radiative transfer equation; Geometrical optics

1. Introduction

The radiative transfer equation (RTE) has the form (see, for example, [1–5] and references within them):

$$\frac{n}{c}\frac{\partial}{\partial t}L(\mathbf{r},\mathbf{\Omega},t)+\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega},t)+[\mu_{\mathrm{a}}(\mathbf{r})+\mu_{\mathrm{s}}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)$$
$$=\mu_{\mathrm{s}}(\mathbf{r})\int_{4\pi}\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)\mathrm{d}\omega'+\varepsilon(\mathbf{r},\mathbf{\Omega},t),\qquad(1)$$

where $L(\mathbf{r},\mathbf{\Omega},t)$ is the radiance at a point $\mathbf{r}=(x_1,x_2,x_3)$ in a direction given by the ray unit vector $\mathbf{\Omega}$, n the refractive index, c the speed of light in vacuum, $\mu_a(\mathbf{r})$ and $\mu_s(\mathbf{r})$ the absorption and scattering coefficients, respectively, $\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')$ the normalized scattering function, $\varepsilon(\mathbf{r},\mathbf{\Omega},t)$ the source distribution per unit volume and ∇_{r} denotes gradient operator with respect to coordinate $\mathbf{r}$.

The RTE (1) is an energy balance equation whose derivation involves the two following assumptions [5]:

(1) The refractive index of the medium is constant.
(2) The divergence of rays is zero ($\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}=0$) everywhere in the medium.

Nevertheless, the RTE is often applied to situations where the refractive index cannot be considered to be constant. For example, it is commonly applied to biological tissues that are composed of several constituent tissue

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types having different values of refractive index [6–10]. The spatial and temporal variation in refractive index is also likely to be dependent on processes occurring in tissue at a cellular level. Consequently, there is clearly a practical need for a generalized RTE which can be applied to media with a spatially varying refractive index, and it would be likely to find important applications in the optics of turbid media, and in biomedical optics in particular.

In 1999 Ferwerda [11] proposed a radiative transfer equation for media with spatially varying refractive index (RTEvri), which was later modified by Khan and Jiang [12] who eliminated a redundant term. The Ferwerda–Khan–Jiang RTEvri has the form:

$$\begin{aligned}\frac{n(\mathbf{r})}{c}\frac{\partial}{\partial t}L(\mathbf{r},\mathbf{\Omega},t)+\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega},t)+[\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ +\nabla_{\mathrm{r}}\ln n(\mathbf{r})\cdot\nabla_{\Omega}L(\mathbf{r},\mathbf{\Omega},t)+[\mu_{\mathrm{a}}(\mathbf{r})+\mu_{\mathrm{s}}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ =\mu_{\mathrm{s}}(\mathbf{r})\int_{4\pi}\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)\mathrm{d}\omega'+\varepsilon(\mathbf{r},\mathbf{\Omega},t),\end{aligned}\tag{2}$$

where $n(\mathbf{r})$ depends on spatial coordinates, ∇_{Ω} denotes gradient with respect to unit vector $\mathbf{\Omega}$ and the ray divergence is given by the expression:

$$\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})=\mathbf{\Omega}^{-1}\cdot\nabla_{\mathrm{r}}\ln n(\mathbf{r})-3\nabla_{\mathrm{r}}\ln n(\mathbf{r})\cdot\mathbf{\Omega}(\mathbf{r}),\tag{3}$$

where $(\Omega_1,\Omega_2,\Omega_3)$ are the components of the unit vector $\mathbf{\Omega}(\mathbf{r})$ on the coordinate axes and $\mathbf{\Omega}^{-1}=(\Omega_1^{-1},\Omega_2^{-1},\Omega_3^{-1})$. Note that for media with constant refractive index the ray divergence (3) equals zero.

Meanwhile, Tualle and Tinet [13] derived a RTEvri with a different form (the Tualle–Tinet RTEvri):

$$\begin{aligned}\frac{n(\mathbf{r})}{c}\frac{\partial}{\partial t}L(\mathbf{r},\mathbf{\Omega},t)+\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega},t)-2[\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathrm{r}}\ln n(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ +\nabla_{\mathrm{r}}\ln n(\mathbf{r})\cdot\nabla_{\Omega}L(\mathbf{r},\mathbf{\Omega},t)+[\mu_{\mathrm{a}}(\mathbf{r})+\mu_{\mathrm{s}}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ =\mu_{\mathrm{s}}(\mathbf{r})\int_{4\pi}\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)\mathrm{d}\omega'+\varepsilon(\mathbf{r},\mathbf{\Omega},t).\end{aligned}\tag{4}$$

In their physical model it is explicitly assumed that the ray divergence is zero everywhere [13]. Shendeleva [14] derived a RTEvri identical to the Tualle–Tinet RTEvri implicitly assuming that the ray divergence is zero.

Another RTEvri was derived by Martí-López et al. [15,16]. The Martí–Bouza–Hebden–Arridge–Martínez RTEvri has the form of the Ferwerda–Khan–Jiang RTEvri (2), but uses a different expression for the ray divergence. It is derived using the relationship between the unit vector $\Omega(\mathbf{r})$ and the eikonal of geometrical optics $\Lambda(\mathbf{r})$ [17]:

$$\mathbf{\Omega}(\mathbf{r})=\frac{\nabla_{\mathrm{r}}\Lambda(\mathbf{r})}{n(\mathbf{r})}.\tag{5}$$

Substituting expression (5) into the ray divergence we obtain their expression for the ray divergence:

$$\begin{aligned}\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})&=\nabla_{\mathrm{r}}\cdot\frac{\nabla_{\mathrm{r}}\Lambda(\mathbf{r})}{n(\mathbf{r})}\\ &=\frac{\nabla_{\mathrm{r}}^2\Lambda(\mathbf{r})}{n(\mathbf{r})}+\nabla_{\mathrm{r}}\Lambda(\mathbf{r})\cdot\nabla_{\mathrm{r}}\frac{1}{n(\mathbf{r})},\qquad(6)\\ &=\frac{\nabla_{\mathrm{r}}^2\Lambda(\mathbf{r})}{n(\mathbf{r})}-\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathrm{r}}\ln n(\mathbf{r}).\qquad(7)\end{aligned}$$

Using a different physical model, Premaratne et al. [18] derived another RTEvri (the Premaratne–Premaratne–Lowery RTEvri):

$$\begin{aligned}\frac{n(\mathbf{r})}{c}\frac{\partial}{\partial t}L(\mathbf{r},\mathbf{\Omega},t)+\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega},t)+[\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ -2[\mathbf{\Omega}(\mathbf{r})\cdot\nabla_{\mathrm{r}}\ln n(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)+\nabla_{\mathrm{r}}\ln n(\mathbf{r})\cdot\nabla_{\Omega}L(\mathbf{r},\mathbf{\Omega},t)\\ +[\mu_{\mathrm{a}}(\mathbf{r})+\mu_{\mathrm{s}}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega},t)\\ =\mu_{\mathrm{s}}(\mathbf{r})\int_{4\pi}\theta(\mathbf{r},\mathbf{\Omega},\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)\mathrm{d}\omega'+\varepsilon(\mathbf{r},\mathbf{\Omega},t),\end{aligned}\tag{8}$$

where the ray divergence is assumed to be [19]

$$\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})=\frac{1}{R_1(\mathbf{r})}+\frac{1}{R_2(\mathbf{r})},\tag{9}$$

where $R_1(\mathbf{r})$ and $R_2(\mathbf{r})$ are the principal radii of curvature of the wavefronts [19].

It is obviously desirable to determine which of these alternative radiative transfer equations gives the best description of light propagation in turbid media with spatially varying refractive index. An evaluation method we propose is to employ them to solve problems of light propagation with well-known solutions, where we assume that a correct RTEvri will retrieve those solutions.

The purpose of this paper is to test the RTE, the Ferwerda–Khan–Jiang RTEvri, the Tualle–Tinet RTEvri, the Martí–Bouza–Hebden–Arridge–Martínez RTEvri and the Premaratne–Premaratne–Lowery RTEvri for a non-absorbing non-scattering isotropic medium ($\mu_{\mathrm{a}}(\mathbf{r})=\mu_{\mathrm{s}}(\mathbf{r})=0$) where the laws of geometrical optics apply, such as the inverse square law and the law of the ratio of irradiances at any two points of a ray. We show that only one of the radiative transfer equations is consistent with these laws. This paper generalizes results reported in a previous paper [16], since validates the terms of the Martí–Bouza–Hebden–Arridge–Martínez RTEvri involving the gradient of refractive index $\nabla_{\mathrm{r}}n(\mathbf{r})$ and includes the study of the Premaratne–Premaratne–Lowery RTEvri.

2. Laws of geometrical optics for testing radiative transfer equations

It is immediately evident that the RTE, the Ferwerda–Khan–Jiang RTEvri, the Tualle–Tinet RTEvri, the Martí–Bouza–Hebden–Arridge-Martínez RTEvri and the Premaratne–Premaratne–Lowery RTEvri have four identical terms as follows:

$$\frac{n(\mathbf{r})}{c}\frac{\partial}{\partial t}L(\mathbf{r},\mathbf{\Omega},t),\tag{10}$$

$$[\mu_a(\mathbf{r}) + \mu_s(\mathbf{r})]L(\mathbf{r}, \mathbf{\Omega}, t), \tag{11}$$

$$\mu_s(\mathbf{r}) \int_{4\pi} \theta(\mathbf{r}, \mathbf{\Omega}, \mathbf{\Omega}')L(\mathbf{r}, \mathbf{\Omega}', t)\mathrm{d}\omega', \tag{12}$$

$$\varepsilon(\mathbf{r}, \mathbf{\Omega}, t), \tag{13}$$

and that they differ in the terms related to the gradient of refractive index $\nabla_r n(\mathbf{r})$, the unit vector $\mathbf{\Omega}$ and its divergence:

$$\nabla_r \ln n(\mathbf{r}) \cdot \nabla_\Omega L(\mathbf{r}, \mathbf{\Omega}, t), \tag{14}$$

$$[\mathbf{\Omega}(\mathbf{r}) \cdot \nabla_r \ln n(\mathbf{r})]L(\mathbf{r}, \mathbf{\Omega}, t), \tag{15}$$

$$[\nabla_r \cdot \mathbf{\Omega}(\mathbf{r})]L(\mathbf{r}, \mathbf{\Omega}, t). \tag{16}$$

Therefore, to compare the above equations, we need to select problems involving non-zero ray divergences and non-zero gradients of refractive index with well-known solutions. Those problems need not involve absorption and scattering effects because all the equations have identical terms for describing them.

Let us consider a case where $\mu_a(\mathbf{r}) = \mu_s(\mathbf{r}) = 0$ and which is time-independent. Thus the terms (10)–(12) of above equations vanish and the resulting equations should describe light propagation in non-absorbing, non-scattering, isotropic media. Therefore the solutions to the correct RTEvri must yield well-known irradiance laws of geometrical optics [16].

Two suitable problems for comparing the RTEvris are the calculation of the irradiance emitted by a time-independent isotropic point source, and the calculation of the ratio of irradiances at two points of a ray, both of which governed by well-known laws of geometrical optics [17]. These laws are now briefly discussed in the following sections.

2.1. The inverse square law

Let us consider a time-independent isotropic point source located at the coordinate origin in a non-absorbing non-scattering isotropic infinite medium of constant refractive index. For such a source the eikonal $\Lambda(\mathbf{r})$, the unit vector of rays $\mathbf{\Omega}(\mathbf{r})$, the ray divergence $\nabla_r \cdot \mathbf{\Omega}(\mathbf{r})$ and the irradiance $I(\mathbf{r})$ are

$$\Lambda(\mathbf{r}) = nr, \tag{17}$$

$$\mathbf{\Omega}(\mathbf{r}) = \frac{\nabla_r \Lambda(\mathbf{r})}{|\nabla_r \Lambda(\mathbf{r})|} = \hat{\mathbf{r}}, \tag{18}$$

$$\nabla_r \cdot \mathbf{\Omega}(\mathbf{r}) = \frac{2}{r} \neq 0, \tag{19}$$

$$I(\mathbf{r}) = \frac{\text{constant}}{r^2}, \tag{20}$$

where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

Eq. (20) is the well-known inverse square law of geometrical optics [17]. Although deriving the inverse square law (20) from the various RTEvri expressions enables them to be tested for the correctness of the treatment of ray divergence, this will not test how useful they are for handling refractive index gradients.

2.2. The law of the ratio of irradiances at any two points of a ray

Let us consider now the time-independent propagation of light in a non-absorbing, non-scattering, isotropic medium with spatially varying refractive index. The ray trajectory is described by the expression:

$$\mathbf{r} = \mathbf{R}(s), \tag{21}$$

where s is the arc length.

For that ray the law of the ratio of irradiances at any two points of a ray holds as follows [17]:

$$\frac{I_2}{I_1} = \frac{n_2}{n_1} \exp\left\{ -\int_{s_1}^{s_2} \frac{\nabla_r^2 \Lambda(s)}{n(s)} \mathrm{d}s \right\}, \tag{22}$$

where $I_1 = I(\mathbf{r}_1)$, $I_2 = I(\mathbf{r}_2)$ are the irradiances at the points $\mathbf{r}_1$ and $\mathbf{r}_2$ of the ray, respectively, $n_1 = n(\mathbf{r}_1)$, $n_2 = n(\mathbf{r}_2)$, $\mathbf{r}_1 = \mathbf{R}(s_1)$, $\mathbf{r}_2 = \mathbf{R}(s_2)$, d$s$ is the differential of arc along the ray trajectory, (see Fig. 1) and the integral is calculated along the ray trajectory.

Deriving the law of the ratio of irradiances at any two points of a ray (22) from the RTEvris allows them to be tested for the treatment of the ray divergence and the treatment of refractive index gradients. Note also that the inverse square law is a particular case of the law of the ratio of irradiances at two points of a ray [17].

3. Testing with the inverse square law

3.1. Testing the RTE, the Ferwerda–Khan–Jiang RTEvri and the Tualle–Tinet RTEvri

In a non-absorbing non-scattering homogeneous infinite medium with a time-independent isotropic point source located at the origin of coordinates the RTE, the Ferwerda–Khan–Jiang RTEvri and the Tualle–Tinet RTEvri are reduced to the equation:

$$\mathbf{\Omega} \cdot \nabla_r L(\mathbf{r}, \mathbf{\Omega}) = \varepsilon(\mathbf{r}), \tag{23}$$

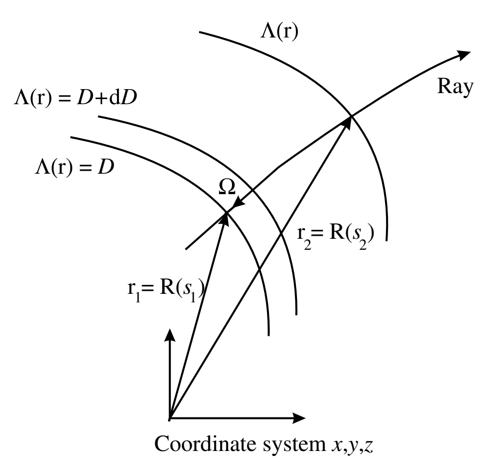

Fig. 1. Ray and wavefronts.

where

$$\varepsilon(\mathbf{r}) = \frac{P_0}{4\pi}\delta(\mathbf{r}), \tag{24}$$

and $P_0 > 0$, $\delta(\mathbf{r})$ are the radiant flux (power) and the Dirac δ function, respectively.

Now we expand the radiance $L(\mathbf{r},\mathbf{\Omega})$ in a series of spherical harmonics of the form:

$$L(\mathbf{r},\mathbf{\Omega}) = \frac{1}{4\pi}I(\mathbf{r}) + \frac{3}{4\pi}\mathbf{\Omega}\cdot J(\mathbf{r}) + R_n(\mathbf{r},\mathbf{\Omega}), \tag{25}$$

where $I(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ are the irradiance and the radiant current density vector, respectively, defined by the expressions:

$$I(\mathbf{r}) = \int_{4\pi} L(\mathbf{r},\mathbf{\Omega})\mathrm{d}\omega, \tag{26}$$

$$\mathbf{J}(\mathbf{r}) = \int_{4\pi} \mathbf{\Omega}(\mathbf{r})L(\mathbf{r},\mathbf{\Omega})\mathrm{d}\omega, \tag{27}$$

and $R_n(\mathbf{r},\mathbf{\Omega})$ represents all the orthogonal terms of higher order.

Substituting expression (25) into Eq. (23), multiplying the resulting equation by unit vector $\mathbf{\Omega}$ and integrating over 4π sr we obtain:

$$\frac{1}{4\pi}\int_{4\pi}\mathbf{\Omega}[\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}I(\mathbf{r})]\mathrm{d}\omega + \frac{3}{4\pi}\int_{4\pi}\mathbf{\Omega}\{\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}[\mathbf{\Omega}\cdot\mathbf{J}(\mathbf{r})]\}\mathrm{d}\omega + \int_{4\pi}\mathbf{\Omega}[\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}R_n(\mathbf{r},\mathbf{\Omega})]\mathrm{d}\omega = \int_{4\pi}\mathbf{\Omega}\varepsilon(\mathbf{r})\mathrm{d}\omega. \tag{28}$$

Applying the following identities [1]:

$$\int_{4\pi}\mathbf{\Omega}\mathrm{d}\omega = 0, \tag{29}$$

$$\int_{4\pi}\mathbf{\Omega}[\mathbf{\Omega}\cdot\mathbf{A}]\mathrm{d}\omega = \frac{4\pi}{3}\mathbf{A}, \tag{30}$$

and the orthogonal properties of the spherical harmonics series to Eq. (28) we obtain:

$$\frac{1}{3}\nabla_{\mathrm{r}}I(\mathbf{r}) = 0 \Rightarrow I(\mathbf{r}) = C. \tag{31}$$

where C is an arbitrary constant.

Therefore, the inverse square law (20) cannot be derived from the RTE, the Ferwerda–Khan–Jiang RTEvri or the Tualle–Tinet RTEvri.

However, it should be pointed out that an inverse square law for the radiant current density vector can be derived from those equations. To demonstrate that, we can add the term $[\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega})$ to the left hand side of expression (23). This does not affect expression (23) because for the RTE, the Ferwerda–Khan–Jiang RTEvri and the Tualle–Tinet RTEvri the ray divergence is $\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r}) = 0$. Substituting expression (24) and vector identity $\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega}) + [\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega}) = \nabla_{\mathrm{r}}\cdot[\mathbf{\Omega}(\mathbf{r})L(\mathbf{r},\mathbf{\Omega})]$ into Eq. (23) we obtain:

$$\nabla_{\mathrm{r}}\cdot[\mathbf{\Omega}(\mathbf{r})L(\mathbf{r},\mathbf{\Omega})] = \frac{P_0}{4\pi}\delta(\mathbf{r}). \tag{32}$$

Integrating Eq. (32) over a sphere of radius r centered at the coordinate origin and applying the Gauss theorem we obtain:

$$\int_S [\mathbf{\Omega}(\mathbf{r})L(\mathbf{r},\mathbf{\Omega})]\cdot\mathrm{d}\mathbf{S} = \frac{P_0}{4\pi}, \tag{33}$$

where S denotes the parametric expression of the surface of the sphere and $\mathrm{d}\mathbf{S}$ is a differential element of the surface oriented outwards.

Integrating Eq. (33) over 4π sr we obtain:

$$\int_S \mathbf{J}(\mathbf{r})\cdot\mathrm{d}\mathbf{S} = P_0. \tag{34}$$

From the symmetrical properties of this problem we observe that $\mathbf{J}(\mathbf{r})\uparrow\uparrow\mathrm{d}\mathbf{S}$ and $|\mathbf{J}(\mathbf{r})| = J(r) = $ constant on the surface S. Applying these properties to Eq. (34) and after simple transformations we obtain:

$$\mathbf{J}(\mathbf{r}) = \frac{P_0}{4\pi r^2}\hat{\mathbf{r}}. \tag{35}$$

The above expression (35) is the inverse square law for the radiant current density vector.

3.2. Testing the Martí–Bouza–Hebden–Arridge–Martínez RTEvri

In a non-absorbing, non-scattering, homogeneous infinite medium with a time-independent isotropic point source located at the coordinate origin the Martí–Bouza–Hebden–Arridge–Martínez RTEvri transforms into the equation:

$$\mathbf{\Omega}\cdot\nabla_{\mathrm{r}}L(\mathbf{r},\mathbf{\Omega}) + [\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}]L(\mathbf{r},\mathbf{\Omega}) = \varepsilon(\mathbf{r}), \tag{36}$$

where $\varepsilon(\mathbf{r})$ is given by expression (24) and the ray divergence $\nabla_{\mathrm{r}}\cdot\mathbf{\Omega}$ by expression (19).

Now we apply the same procedures we employed earlier to obtain expressions (31) and (35). This yields

$$\frac{1}{3}\nabla_{\mathrm{r}}I(\mathbf{r}) = -\frac{2}{r}\mathbf{J}(\mathbf{r}), \tag{37}$$

$$\mathbf{J}(\mathbf{r}) = \frac{P_0}{4\pi r^2}\hat{\mathbf{r}}. \tag{38}$$

Substituting Eq. (38) into Eq. (37), applying the symmetrical properties to the resulting equation and rewriting it in spherical coordinates we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}r}I(r) = -\frac{3P_0}{2\pi r^3}, \tag{39}$$

The solution to this elementary differential equation is

$$I(r) = \frac{3P_0}{4\pi r^2} + C, \tag{40}$$

where C is a constant.

The condition that the irradiance approaches zero as r approaches infinity requires that $C = 0$. Thus we obtain:

$$I(r) = \frac{3P_0}{4\pi r^2}, \tag{41}$$

which is consistent with the inverse square law (20).

Note that for the Martí-Bouza–Hebden–Arridge–Martínez RTEvri the expressions Eqs. (32)–(35) also hold. Therefore, the inverse square law for the radiant current density vector can also be derived from the Martí–Bouza–Hebden–Arridge–Martínez RTEvri.

3.3. Testing the Premaratne–Premaratne–Lowery RTEvri

Premaratne et al. [18] assumed that in a medium of constant refractive index the wavefronts are always plane surfaces. Therefore, the ray divergence is zero ($\nabla_r \cdot \Omega = 0$), the Premaratne–Premaratne–Lowery RTEvri reduces to the form shown in Eq. (23) and the inverse square law cannot be derived from the Premaratne–Premaratne–Lowery RTEvri (see Section 3.1). This result is not a consequence of an incorrect expression for the ray divergence, but due to the assumption of plane waves propagating in the homogenous medium. To prove that, we can consider wavefronts emerging from an isotropic point source in homogeneous medium as spherical surfaces with principal curvature radii:

$$R_1(\mathbf{r}) = R_2(\mathbf{r}) = r. \tag{42}$$

Substituting the expression (42) into the ray divergence (9) we obtain:

$$\nabla_r \cdot \mathbf{\Omega} = \frac{2}{r}. \tag{43}$$

Note that the above expression is identical to Eq. (19), as we should expect. Repeating the same procedure employed to obtain expressions Eqs. (38) and (41), we can show that the inverse square law can be derived from the Premaratne–Premaratne–Lowery RTEvri.

4. Testing with the law of the ratio of irradiances at two points of a ray

4.1. Testing the RTE, the Ferwerda–Khan–Jiang RTEvri and the Tualle–Tinet RTEvri

Since the inverse square law cannot be derived from the RTE, the Ferwerda–Khan–Jiang RTEvri or the Tualle–Tinet RTEvri, it follows that none of these can be used to derive the more general law.

4.2. Testing the Martí-Bouza–Hebden–Arridge–Martínez RTEvri

In a non-absorbing, non-scattering, isotropic non-homogeneous medium, with no source the time-independent Martí–Bouza–Hebden–Arridge–Martínez RTEvri has the following the form:

$$\mathbf{\Omega} \cdot \nabla_r L(\mathbf{r},\mathbf{\Omega}) + [\nabla_r \cdot \mathbf{\Omega}] L(\mathbf{r},\mathbf{\Omega}) + \nabla_r \ln n(\mathbf{r}) \cdot \nabla_\Omega L(\mathbf{r},\mathbf{\Omega}) = 0, \tag{44}$$

Substituting expression (6) into Eq. (44) and arranging the resulting equation we obtain:

$$\frac{\nabla_r \Lambda(\mathbf{r})}{n(\mathbf{r})} \cdot \nabla_r \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} + \nabla_\Omega \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} \cdot \frac{\nabla_r n(\mathbf{r})}{n(\mathbf{r})} = -\frac{\nabla_r^2 \Lambda(\mathbf{r})}{n^2(\mathbf{r})} L(\mathbf{r},\mathbf{\Omega}), \tag{45}$$

Now we apply Eq. (45) along the ray trajectory $\mathbf{r} = \mathbf{R}(s)$ between the wavefronts $\Lambda(\mathbf{r}) = D$ and $\Lambda(\mathbf{r}) = D + \mathrm{d}D$, where D is a real number and $\mathrm{d}D$ a differential increment (see Fig. 1). Along the ray trajectory the following formulae hold:

$$\frac{\nabla_r \Lambda(\mathbf{r})}{n(\mathbf{r})} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}, \tag{46}$$

$$\nabla_\Omega \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} \cdot \frac{\nabla_r n(\mathbf{r})}{n(\mathbf{r})} = \nabla_\Omega \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} \cdot \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}s}, \tag{47}$$

Identity (46) derives from the general properties of rays and wavefronts [17] and identity (47) from the properties of operator ∇_Ω [12]. Substituting formulae (46) and (47) into Eq. (45) and using the chain rule we obtain:

$$\nabla_r \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} + \nabla_\Omega \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} \cdot \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s} \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s} \frac{L(\mathbf{r},\mathbf{\Omega})}{n(\mathbf{r})} = -\frac{\nabla_r^2 \Lambda(\mathbf{r})}{n^2(\mathbf{r})} L(\mathbf{r},\mathbf{\Omega}), \tag{48}$$

Integrating Eq. (48) over 4π sr we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{I(s)}{n(s)} = -\frac{\nabla_r^2 \Lambda(s)}{n^2(s)} I(s) \tag{49}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s} \ln\left[\frac{I(s)}{n(s)}\right] = -\frac{\nabla_r^2 \Lambda(s)}{n(s)}, \tag{50}$$

where $I(s) = I[\mathbf{R}(s)]$, $n[\mathbf{R}(s)] = n(s)$ and $\Lambda[\mathbf{R}(s)] = \Lambda(s)$.

A further integration of expression (50) along the ray trajectory yields:

$$\ln\left[\frac{I(s_2)}{n(s_2)}\right] - \ln\left[\frac{I(s_1)}{n(s_1)}\right] = \int_{s_1}^{s_2} -\frac{\nabla_r^2 \Lambda(s)}{n(s)} \mathrm{d}s. \tag{51}$$

After a simple transformation of Eq. (51) we can obtain expression (22). Therefore, the law of the ratio of irradiances at two points of a ray can be derived from the Martí–Bouza–Hebden–Arridge–Martínez RTEvri.

4.3. Testing the Premaratne–Premaratne–Lowery RTEvri

In a non-absorbing, non-scattering, isotropic non-homogeneous medium with no source, the time-independent Premaratne–Premaratne–Lowery RTEvri adopts the form:

$$\mathbf{\Omega} \cdot \nabla_r L(\mathbf{r},\mathbf{\Omega}) + [\nabla_r \cdot \mathbf{\Omega}(\mathbf{r})] L(\mathbf{r},\mathbf{\Omega}) - 2[\mathbf{\Omega}(\mathbf{r}) \cdot \nabla_r \ln n(\mathbf{r})] L(\mathbf{r},\mathbf{\Omega})$$
$$+ \nabla_r \ln n(\mathbf{r}) \cdot \nabla_\Omega L(\mathbf{r},\mathbf{\Omega}) = 0, \tag{52}$$

where the ray divergence is given by expression (9).

From a comparison between expression (52) and expression (44) it is evident that the former has an additional term $-2[\mathbf{\Omega}(\mathbf{r})\cdot\nabla_r \ln n(\mathbf{r})]L(\mathbf{r},\mathbf{\Omega})$, which does not allow the law of the ratio of irradiances at two points of a ray (22) to be derived.

5. Conclusion

We demonstrated that the inverse square law of geometrical optics cannot be derived from the RTE, the Ferwerda–Khan–Jiang RTEvri or the Tualle–Tinet RTEvri. In contrast, it can be derived from the Martí-Bouza–Hebden–Arridge–Martínez RTEvri or the Premaratne–Premaratne–Lowery RTEvri. This is due to the ability of the latter equations to accommodate the ray divergence $\nabla_r\cdot\mathbf{\Omega}(\mathbf{r})$ and confirms an earlier result [16]. We also demonstrated that the law of the ratio of irradiances at two points of a ray can be derived from the Martí–Bouza–Hebden–Arridge–Martínez RTEvri. Therefore, the Martí–Bouza–Hebden–Arridge–Martínez RTEvri not only correctly models the ray divergence, but also the effect of gradients of refractive index.

The significance of our result is manifold. First, it provides further support to the hypothesis that the failure of the diffusion equation (DE) in the vicinity of an isotropic point source can be interpreted as due to an inherent assumption of zero ray divergence in the RTE, a condition that is not met within that region [16]. Second, in the so-called void regions, where the scattering coefficient is small and the RTE and the DE fail, other models of light propagation as the hybrid radiosity-diffusion theory, have been proposed [20,21]. In this regard the Martí–Bouza–Hebden–Arridge–Martínez RTEvri (or derived equations) could be an alternative to the radiosity-diffusion theory, with the advantage that the former is linked with geometrical optics. For the first time, we present an analytical model which can accommodate the fundamental principles of both radiative transfer theory and geometrical optics.

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